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On a Type of Spacetime

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Abstract. The object of the present paper is to study a special type of spacetime. It is proved that in a conformally flat $(WRS)_4$ spacetime with non-zero scalar curvature the vector field ρ defined by $g(X, \rho) = E(X)$ is irrotational and the integral curves of the vector field ρ are geodesics. We also show that a conformally flat $(WRS)_4$ spacetime with non-zero scalar curvature is the Robertson-Walker spacetime. Next possible local cosmological structure of such a spacetime is determined. Finally, we construct an example of a conformally flat $(WRS)_4$ spacetime with non-zero scalar curvature.

1. Introduction

The present paper is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold (M^4, g) with Lorentz metric g with signature $(-,+,+,+)$. The geometry of the Lorentz manifold begins with the study of the causal character of vectors of the manifold. It is due to this causality that the Lorentz manifold becomes a convenient choice for the study of general relativity.

Here we consider a special type of spacetime which is called conformally flat weakly Ricci symmetric spacetime. A Riemannian manifold is said to be Ricci symmetric if its Ricci tensor $S \neq 0$ and satisfies the condition $\nabla S = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. In 1993 Tamássy and Binh [22] introduced the notion of a weakly Ricci symmetric manifold. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called weakly Ricci symmetric if its Ricci tensor S is of type $(0, 2)$ satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + D(Z)S(Y, X), \tag{1}$$

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where A, B, D are three non-zero 1-forms. Such an n-dimensional manifold was denoted by $(WRS)_n$. If $A = B = D = 0$, then the manifold reduces to a Ricci symmetric ($\nabla S = 0$) manifold. If in (1) the 1-form A is replaced by $2A$, B and D are replaced by A, then the manifold is called a pseudo Ricci symmetric manifold introduced by Chaki [2]. Also if in (1) the 1-form A is replaced by $2A$, then the manifold is called a generalized pseudo Ricci symmetric manifold introduced by Chaki and Koley [3]. So the defining condition of a $(WRS)_n$ is little weaker than that of a generalized pseudo Ricci symmetric manifold. In a recent paper De and Ghosh [6] cited an example of a $(WRS)_n$. In this connection we mention the works of Defever et al. [8], Shaikh et al. [20], Kaigorodov [12], De and Ghosh [5], Chaki and Roy [4], De et al. [7], Guha and Chakraborty [11], Prvanovic [16, 17], Tamássy and Binh [21] and many others.
This paper is organized as follows: After preliminaries, in Section 3, we show that in a conformally flat $(WRS)_4$ spacetime with non-zero scalar curvature the vector field ρ defined by $g(X, \rho) = E(X)$ is irrotational and the integral curves of the vector field ρ are geodesics and this spacetime is the Robertson-Walker spacetime. In this Section we also prove that in a perfect fluid conformally flat $(WRS)_4$ with non-zero scalar curvature the fluid has vanishing vorticity and vanishing shear. Next we prove that if in a conformally flat $(WRS)_4$ perfect fluid spacetime with non-zero scalar curvature the velocity vector field is always hypersurface orthogonal, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O. Finally, we construct an example of a conformally flat $(WRS)_4$ spacetime with non-zero scalar curvature.

2. Preliminaries

In the study of a $(WRS)_4$ spacetime an important role is played by the 1-form δ defined by

$$\delta(X) = B(X) - D(X).$$

Lemma 2.1 ([5]). *If $\delta \neq 0$, then the Ricci tensor is of the form*

$$S(X, Y) = rE(X)E(Y), \tag{2}$$

where E is a non-zero 1-form defined by

$$E(X) = g(X, \rho), \tag{3}$$

r is the scalar curvature and ρ is called the basic vector field of $(WRS)_4$.

Proof. Let L denote the symmetric endomorphism of the tangent space of a $(WRS)_4$ at each point corresponding to the Ricci tensor S. Then

$$g(LX, Y) = S(X, Y),$$

for all vector fields X, Y.
From (1) it follows that

$$(\nabla_X S)(Y,Z) - (\nabla_X S)(Z,Y) = [B(Y) - D(Y)]S(X,Z) + [D(Z) - B(Z)]S(X,Y), \tag{4}$$

or

$$[B(Y) - D(Y)]S(X,Z) = [B(Z) - D(Z)]S(X,Y), \tag{5}$$

since S is symmetric.
Since $\delta(X) = B(X) - D(X)$, the equation (5) becomes

$$\delta(Y)S(X,Z) = \delta(Z)S(X,Y). \tag{6}$$

Putting $X = Z = e_i$ in (6) and taking summation over i, $1 \leq i \leq 4$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get

$$\delta(Y)r = \delta(LY), \tag{7}$$

where $\delta(X) = g(X,\nu)$ for every vector field X and r is the scalar curvature.
From (6) we have

$$\delta(\nu)S(X,Z) = \delta(Z)S(X,\nu) = \delta(Z)g(LX,\nu) = \delta(Z)\delta(LX).$$

Hence using (7) we have

$$S(X,Z) = r\frac{\delta(X)}{\sqrt{\delta(\nu)}}\frac{\delta(Z)}{\sqrt{\delta(\nu)}}. \tag{8}$$

Suppose $E(X) = \frac{\delta(X)}{\sqrt{\delta(\nu)}}$ and $g(X,\rho) = E(X)$, ρ is a unit vector. Then (8) can be written as

$$S(X,Z) = rE(X)E(Z),$$

which completes the proof of the Lemma. □

A semi-Riemannian $(WRS)_4$ may similarly be defined by taking a Lorentz metric g with signature $(-,+,+,+)$. All the above relations will also hold in such a $(WRS)_4$.
Now we take ρ as a timelike vector field. Then we have from (2)

$$\begin{aligned} & S(X,Y) = rE(X)E(Y), \\ \text{or} \quad & S(X,\rho) = -rE(X), \quad \text{since} \quad E(\rho) = g(\rho,\rho) = -1, \\ \text{or} \quad & S(X,\rho) = -rg(X,\rho). \end{aligned} \tag{9}$$

3. Conformally flat Weakly Ricci symmetric spacetimes

In this section we consider conformally flat $(WRS)_4$ spacetime with non-zero scalar curvature. Then $divC = 0$, where 'C' denotes the Weyl's conformal curvature tensor and 'div' denotes divergence.
Hence we have [9]

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = \frac{1}{6}\big[g(Y,Z)dr(X) - g(X,Y)dr(Z)\big]. \tag{10}$$

Equation (2) implies

$$\begin{aligned}(\nabla_Z S)(X,Y) = {} & dr(Z)E(X)E(Y) \\ & + r\big[(\nabla_Z E)(X)E(Y) + E(X)(\nabla_Z E)(Y)\big].\end{aligned} \tag{11}$$

Substituting (11) in (10) we get

$$\begin{aligned} & dr(X)E(Z)E(Y) + r\big[(\nabla_X E)(Z)E(Y) + E(Z)(\nabla_X E)(Y)\big] \\ & - dr(Z)E(Y)E(Z) - r\big[(\nabla_Z E)(Y)E(X) + E(Y)(\nabla_Z E)(X)\big] \\ = & \frac{1}{6}\big[g(Y,Z)dr(X) - g(X,Y)dr(Z)\big].\end{aligned} \tag{12}$$

Let $\{e_i\}$ $(i = 1,2,3,4)$ be an orthonormal basis of the tangent space at each point of the spacetime. Setting $Y = Z = e_i$ in (12) and taking summation for $1 \leq i \leq 4$, we get

$$-\frac{3}{2}dr(X) = dr(\rho)E(X) + r\epsilon_i(\nabla_{e_i} E)(e_i)E(X) + r(\nabla_\rho E)(X), \tag{13}$$

where $\epsilon_i = g(e_i, e_i)$, since ρ is a unit timelike vector field, that is, $g(\rho,\rho) = -1$ and $(\nabla_X E)(\rho) = 0$.
Now setting $Y = Z = \rho$ in (12) and using $(\nabla_X E)(\rho) = 0$ and $g(\rho,\rho) = E(\rho) = -1$, we get

$$\frac{7}{6}dr(X) + \frac{7}{6}dr(\rho)E(X) = -r(\nabla_\rho E)(X). \tag{14}$$

Substituting (14) in (13), we get

$$\frac{1}{3}dr(X) = -\frac{1}{6}dr(\rho)E(X) + r\epsilon_i(\nabla_{e_i} E)(e_i)E(X). \tag{15}$$

Putting $X = \rho$ in (15) yields

$$\frac{1}{2}dr(\rho) = -r\epsilon_i(\nabla_{e_i} E)(e_i). \tag{16}$$

From (15) and (16) it follows that

$$dr(X) = -2dr(\rho)E(X). \tag{17}$$

Setting $Y = \rho$ in (12) and using (17), we obtain

$$(\nabla_Z E)(X) - (\nabla_X E)(Z) = 0, \tag{18}$$

for all X, Z, which implies that the 1-form E is closed.

Hence it follows that

$$g(\nabla_X \rho, Z) = g(\nabla_Z \rho, X), \tag{19}$$

which means that the vector field ρ is irrotational.

Now putting $Z = \rho$ in (19) we get

$$g(\nabla_X \rho, \rho) = g(\nabla_\rho \rho, X). \tag{20}$$

Since $g(\nabla_X \rho, \rho) = 0$, from (20) it follows that $g(\nabla_\rho \rho, X) = 0$ for all X. Hence $\nabla_\rho \rho = 0$. This means that the integral curves of the vector field ρ are geodesic.

Thus we can state the following:

Theorem 3.1. *In a conformally flat $(WRS)_4$ with non-zero scalar curvature the vector field ρ defined by* (3) *is irrotational and the integral curves of the vector field ρ are geodesics.*

We now consider the scalar function $f = \frac{1}{6}\frac{dr(\rho)}{r}$.

We have

$$\nabla_X f = \frac{1}{6}\frac{dr(\rho)}{r^2} dr(X) + \frac{1}{6r} d^2 r(\rho, X). \tag{21}$$

On the other hand, (17) implies

$$d^2 r(Y, X) = -2[d^2 r(\rho, Y)E(X) + dr(\rho)(\nabla_X E)Y]$$

from which we get by using (18)

$$d^2 r(\rho, Y)E(X) = d^2 r(\rho, X)E(Y). \tag{22}$$

Putting $X = \rho$ in (22) it follows that

$$d^2 r(\rho, Y) = -d^2 r(\rho, \rho)E(Y) = hE(Y),$$

since $E(\rho) = -1$ and where h is a scalar function.

Thus

$$\nabla_X f = \mu E(X), \tag{23}$$

where $\mu = \frac{1}{6r}[h + \frac{dr(\rho)}{r} dr(\rho)]$, using (17).

Using (23) it is easy to show that

$$\omega(X) = \frac{1}{6}\frac{dr(\rho)}{r} E(X) = fE(X)$$

is closed.

Using (17) and (18) in (12) we get

$$r[E(Z)(\nabla_X E)(Y) - E(X)(\nabla_Z E)(Y)] = \frac{dr(\rho)}{6}[g(Y, Z)E(X) - g(X, Y)E(Z)]$$

Now putting $Z = \rho$ in the above expression yields

(24)

$$(\nabla_X E)(Y) = \frac{1}{6}\frac{dr(\rho)}{r}[E(X)E(Y) + g(X, Y)] = f[E(X)E(Y) + g(X, Y)].$$

Thus, (24) can be written as follows:

$$(\nabla_X E)(Y) = fg(X,Y) + \omega(X)E(Y), \tag{25}$$

where ω is closed. But this means that the vector field ρ corresponding to the 1-form E defined by $g(X,\rho) = E(X)$ is a proper concircular vector field [19, 23].
K. Yano [24] proved that in order that a Riemannian space admits a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

$$ds^2 = (dx^1)^2 + e^q g^*_{\alpha\beta} dx^\alpha dx^\beta, \tag{26}$$

where $g^*_{\alpha\beta} = g^\delta_{\alpha\beta}(x^\gamma)$ are the function of x^γ only $(\alpha,\beta,\gamma,\delta = 2,3,....,n)$ and $q = q(x^1) \neq$ constant is a function of x^1 only. Similarly, we can prove that a Lorentzian space with the metric of signature $(-,+,+,+)$ admits a concircular vector field if and only if there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

$$ds^2 = -(dx^1)^2 + e^q g^*_{\alpha\beta} dx^\alpha dx^\beta. \tag{27}$$

Then if a $(WRS)_4$ spacetime is conformally flat, that is, if it satisfies (10), it is a warped product $-I \times_{e^q} M^*$, where (M^*, g^*) is a 3-dimensional manifold. A. Gebarowski [10] proved that the warped product $I \times_{e^q} M^*$ satisfies (10) if and only if M^* is an Einstein manifold. Thus if $(WRS)_4$ satisfies (10), it must be a warped product $-I \times_{e^q} M^*$, where M^* is an Einstein manifold. But a 3-dimensional Einstein manifold is a manifold of constant curvature. Hence a conformally flat $(WRS)_4$ can be expressed as a warped product $-I \times_{e^q} M^*$, M^* is a manifold of constant curvature. But such a warped product is the Robertson-Walker spacetime. Hence we conclude the following:

Theorem 3.2. *A conformally flat $(WRS)_4$ spacetime with non-zero scalar curvature is the Robertson-Walker spacetime.*

Finally, we consider conformally flat $(WRS)_4$ perfect fluid spacetime. Then the energymomentum tensor T is of the form [13, 14]

$$T(X,Y) = (p+\sigma)E(X)E(Y) + pg(X,Y),$$

where σ is the energy density and p is the isotropic pressure of the fluid. The velocity vector field ρ of the fluid corresponding to the 1-form E is a timelike vector field. We assume that the velocity vector field of the fluid is hypersurface orthogonal and the energy density is constant over a hypersurface orthogonal to ρ. From Theorem 3.1 we get that the integral curves of ρ in a conformally flat spacetime are geodesics, the Roy Choudhury equation [18] for the fluid can be written as

$$(\nabla_X E)(Y) = \tilde{\omega}(X,Y) + \tau(X,Y) + f[g(X,Y) + E(X)E(Y)], \tag{28}$$

where $\tilde{\omega}$ is the vorticity tensor and τ is the shear tensor respectively.
Comparing (24) and (28) we get

$$\tilde{\omega}(X,Y) + \tau(X,Y) = 0. \tag{29}$$

Again from Theorem 3.1 it follows that ρ is irrotational. Hence the vorticity of the fluid vanishes. Therefore $\tilde{\omega}(X,Y) = 0$ and consequently (29) implies that $\tau(X,Y) = 0$. Thus we can state the following:

Theorem 3.3. *In a perfect fluid conformally flat $(WRS)_4$ spacetime with non-zero scalar curvature, the fluid has vanishing vorticity and vanishing shear.*

According to Petrov classification a spacetime can be devided into six types denoted by I, II, III, D, N and O [15]. Again Barnes [1] has proved that if a perfect fluid spacetime is shear free, vorticity free and the velocity vector field is hypersurface orthogonal and the energy density is constant over a hypersurface orthogonal to the velocity vector field, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O. Thus from Theorem 3.3 we can state the following:

Theorem 3.4. *If in a conformally flat $(WRS)_4$ perfect fluid spacetime with non-zero scalar curvature the velocity vector field is always hypersurface orthogonal, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O.*

4. Example of a conformally flat $(WRS)_4$ spacetime

In this section, we prove the existence of a conformally flat $(WRS)_4$ spacetime with non-zero non-constant scalar curvature by constructing a non-trivial example.
Let us consider a Lorentzian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2], \tag{30}$$

where $q = \frac{e^{x^1}}{k^2}$ and k is a non-zero constant, $(i,j = 1,2,3,4)$. Here the signature of g is $(+,+,+,-)$ which is Lorentzian. Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are:

$$\Gamma^1_{11} = \Gamma^1_{44} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{q}{1+2q}, \qquad \Gamma^1_{22} = \Gamma^1_{33} = -\frac{q}{1+2q},$$

$$R_{1221} = R_{1331} = \frac{q}{1+2q}, \qquad R_{1441} = -\frac{q}{1+2q},$$

$$R_{2332} = \frac{q^2}{1+2q}, \qquad R_{2442} = R_{3443} = -\frac{q^2}{1+2q}$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors and their covariant derivatives are:
(31)

$$R_{11} = \frac{3q}{(1+2q)^2}, \qquad R_{22} = R_{33} = \frac{q}{1+2q}, \qquad R_{44} = -\frac{q}{1+2q}$$
$$R_{11,1} = \frac{3q(1-2q)}{(1+2q)^3}, \qquad R_{22,1} = R_{33,1} = \frac{q}{(1+2q)^2},$$
$$R_{44,1} = -\frac{q}{(1+2q)^2}.$$

It can be easily shown that the scalar curvature r of the resulting space $(\mathbb{R}^4, g)$ is $r = \frac{6q(1+q)}{(1+2q)^3}$, which is non-vanishing and non-constant. We easily show that $\mathbb{R}^4$ is conformally flat. We shall now show that $\mathbb{R}^4$ is a coformally flat $(WRS)_4$ spacetime. Let us choose the associated 1-forms as

$$A_i(x) = \begin{cases} \frac{1}{1+2q} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{32}$$

$$B_i(x) = \begin{cases} -\frac{4q}{1+2q} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{33}$$

$$D_i(x) = \begin{cases} \frac{2q}{1+2q} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{34}$$

at any point $x \in \mathbb{R}^4$. With these 1-forms we can easily check that $(\mathbb{R}^4, g)$ is a conformally flat $(WRS)_4$ spacetime whose scalar curvature is non-zero and non-constant. Thus we can state the following:

Theorem 4.1. *Let $(\mathbb{R}^4, g)$ be a 4-dimensional Lorentzian space endowed with the Lorentzian metric g given by*

$$ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2],$$

where $q = \frac{e^{x^1}}{k^2}$ and k is a non-zero constant, $(i, j = 1, 2, 3, 4)$. Then $(\mathbb{R}^4, g)$ is a conformally flat $(WRS)_4$ spacetime with non-zero and non-constant scalar curvature.

5. CONCLUSION

In general relativity the matter content of the spacetime is described by the energy momentum tensor T which is to be determined from physical considerations dealing with the distribution of matter and energy. Since the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological models, the physical motivation for studying Lorentzian manifolds is the assumption that a gravitational field may be

effectively modeled by some Lorentzian metric defined on a suitable four dimensional manifold M. The Einstein equations are fundamental in the construction of cosmological models which imply that the matter determines the geometry of the spacetime and conversely the motion of matter is determined by the metric tensor of the space which is non-flat. Relativistic fluid models are of considerable interest in several areas of astrophysics, plasma physics and nuclear physics. Theories of relativistic stars (which would be models for supermassive stars) are also based on relativistic fluid models. The problem of accretion onto a neutron stars or a blackhole is usually set in the framework of relativistic fluid models.
The physical motivation for studying various types of spacetime models in cosmology is to obtain the information of different phases in the evolution of the universe, which may be classified into three phases, namely, the initial phase, the intermediate phase and the final phase. In the present paper it is shown that a conformally flat $(WRS)_4$ spacetime with non-zero scalar curvature is the Robertson-Walker spacetime. This means that the spacetime is homogeneous and isotropic. Also we prove that if in a conformally flat $(WRS)_4$ perfect fluid spacetime with non-zero scalar curvature the velocity vector field is always hypersurface orthogonal, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O. Finally, we construct an example of a conformally flat $(WRS)_4$ spacetime with non-zero scalar curvature.

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